AXISYMMETRIC CONTACT PROBLEM FOR AN ELASTIC LAYER

(OSESIMMETRICHNAIA KONTAKTNAIA ZADACHA DLIA UPRUGOGO SLOIA)

PMM Vol.22, No.3, 1958, pp. 320-326

N.N. LEBEDEV and Ia.S. UFLIAND
(Leningrad)

(Received 3 July 1957)

The solution of the contact problems of the theory of elasticity is usually based upon the assumption that the body acted upon by a rigid stamp is an elastic half-space. The present paper deals with the more complicated problem of pressing a stamp of circular cross-section into an elastic layer. The method developed in the paper permits to express the required displacements and stresses in terms of one auxiliary function, which represents the solution of a Fredholm integral equation with a continuous symmetrical kernel. A series of numerical results is given for the case of a stamp with plane base.

1. We consider the state of elastic equilibrium of an infinite layer, resting on a rigid immovable basis and undergoing deformation under the

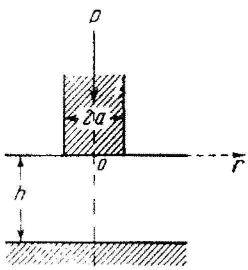

Fig. 1.

action of a rigid stamp of circular cross-section. It is assumed that the contact surface of the stamp is a surface of revolution and that the pressing is realized by means of an axial force (Fig.1). It is furthermore assumed that there is no friction between the stamp and the layer, although the method offered in the following permits generalization to cases of more involved nature (e.g. it is possible to consider layer and basis to be in conditions of adhesion).

Under the assumptions just stated the contact problem can be reduced to the integration of equations of the theory of elasticity in cylindrical coordinates r, ϕ, z with the mixed boundary conditions

$$\tau_{rz} = 0, \quad w = 0 \qquad \text{when } z = h \tag{1.1}$$

$$\tau_{rz} = 0, \quad w = w_0 - \chi(r)\,(r < a), \quad \sigma_z = 0 \ (r > a) \qquad \text{when } z = 0 \tag{1.2}$$

where $(u,\ 0,\ w)$ are the components of the displacement vector in a system of cylindrical coordinates, while σ_z and τ_{rz} are the normal and the tangential stress components, respectively, $\chi(r)$ the curve determining the shape of the pressing surface of the stamp and w_0 the displacement of the latter in the z-direction.

The radius a of the stamp is considered to be given, which is, in the case of a curved shape of the pressing surface of the stamp, equivalent to the case of a complete penetration under the action of a sufficiently large force P. In the case of an incomplete penetration the problem becomes more complicated insofar as the radius of the boundary circle of the contact surface is unknown, so that it must be determined from the continuity condition of the normal stresses at the points of the circle $r = a$.

The solution of the problem under consideration is facilitated by the use of the harmonic functions of Papkovich and Neuber for the representation of the components of the displacement vector. In our case the expressions in question assume the form

$$2\mu u = -\frac{\partial \Phi}{\partial r}, \qquad 2\mu w = -\frac{\partial \Phi}{\partial z} + 4(1-\nu)\,\Phi_1, \qquad \Phi = \Phi_0 + z\Phi_1 \tag{1.3}$$

where μ is the shear modulus, while ν is Poisson's ratio; Φ_0 and Φ_1 are functions, harmonic in the layer $0 < z < h$.

Using the relations (1.1) to (1.3), as well as the formulas

$$\sigma_z = 2(1-\nu)\frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial z} - z\,\frac{\partial^2 \Phi_1}{\partial z^2}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left[(1-2\nu)\,\Phi_1 - \Phi_2 - z\,\frac{\partial \Phi_1}{\partial z}\right], \qquad \Phi_2 = \frac{\partial \Phi_0}{\partial z} \tag{1.4}$$

by which the elastic stresses are expressed in terms of the functions just introduced, we arrive at the boundary conditons

$$[\Phi_1]_{z=h}, \qquad \left[\Phi_2 + h\,\frac{\partial \Phi_1}{\partial z}\right]_{z=h} = 0 \tag{1.5}$$

$$[(1-2\nu)\Phi_1-\Phi_2]_{z=0}=0 \tag{1.6}$$

$$[(3-4\nu)\Phi_1-\Phi_2]_{z=0}=2\mu\,[w_0-\chi(r)] \quad \text{when } r<a \tag{1.7}$$

$$\left[2(1-\nu)\frac{\partial\Phi_1}{\partial z}-\frac{\partial\Phi_2}{\partial z}\right]_{z=0}=0 \quad \text{when } r>a \tag{1.8}$$

These conditions lead to a single-valued determination of Φ_0 and Φ_1. It is assumed that at $r\to\infty$ the functions Φ_1 and Φ_2 are of the order $O(r^{-1})$ and their derivatives tend toward $O(r^{-2})$ at the same limit; this assures the necessary behavior of displacements and stresses at infinity.

2. The solution of the formulated problem is based on its reduction to paired integral equations, which admit an exact solution in quadratures by means of an auxiliary function satisfying Fredholm's integral equation with a continuous symmetrical kernel.

We seek the harmonic functions Φ_1 and Φ_2 in the form

$$\begin{aligned}\Phi_1&=\int_0^\infty A(\lambda)\operatorname{sh}\lambda(h-z)\,J_0(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h}\\ \Phi_2&=\int_0^\infty[\lambda hA(\lambda)\operatorname{ch}\lambda(h-z)+B(\lambda)\operatorname{sh}\lambda(h-z)]\,J_0(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h}\end{aligned} \tag{2.1}$$

where $A(\lambda)$ and $B(\lambda)$ are functions to be determined, while $J_0(x)$ is a Bessel function.

If the functions Φ_1 and Φ_2 are selected in this manner, the boundary conditions (1.5) will be satisfied for any values of $A(\lambda)$ and $B(\lambda)$. Of the remaining conditions the equation (1.6) permits to derive the relation

$$B(\lambda)=(1-2\nu-\lambda h\operatorname{cth}\lambda h)\,A(\lambda) \tag{2.2}$$

while the conditions (1.7) and (1.8) lead to a system of two integral equations for $A(\lambda)$:

$$\int_0^\infty A(\lambda)J_0(\lambda r)\,d\lambda=f(r)\quad(r<a),\qquad \int_0^\infty\frac{\lambda A(\lambda)}{1-g(\lambda)}J_0(\lambda r)\,d\lambda=0\quad(r>a) \tag{2.3}$$

where

$$g(\lambda)=\frac{\lambda h+\operatorname{sh}\lambda h e^{-\lambda h}}{\lambda h+\operatorname{sh}\lambda h\operatorname{ch}\lambda h},\qquad f(r)=\frac{\mu}{1-\nu}[w_0-\chi(r)] \tag{2.4}$$

The solution of equations (2.3) is again sought in the form*

$$A(\lambda)=[1-g(\lambda)]\int_0^a\varphi(t)\cos\lambda t\,dt \tag{2.5}$$

* See [2], where an analogous method was used for the solution of a problem in electrostatics.

where $\phi(t)$ is some unknown function, continuous, together with its derivative, in the closed interval $(0, a)$.

Integrating the right-hand member of (2.5) by parts and substituting* the resulting expression for $A(\lambda)$ into the second equation of the system (2.3), we obtain, for $r > a$,

$$\int_0^\infty \frac{\lambda A(\lambda)}{1-g(\lambda)} J_0(\lambda r)\, d\lambda = \varphi(a) \int_0^\infty J_0(\lambda r) \sin \lambda a\, d\lambda - \int_0^a \varphi'(t)\, dt \int_0^\infty J_0(\lambda r) \sin \lambda t\, d\lambda = 0$$

Since, according to a known formula,

$$\int_0^\infty J_0(\lambda r) \sin \lambda t\, d\lambda = \begin{cases} 0 & (0 \leqslant t < r) \\ (t^2 - r^2)^{-1/2} & (t > r) \end{cases} \tag{2.6}$$

the integrals with respect to the variable λ are zero.

Thus one of the paired integral equations (2.3) is satisfied identically.

Substitution of $A(\lambda)$ into the first of the equations (2.3) after change of order of integration and use of the formulas

$$\int_0^\infty J_0(\lambda r) \cos \lambda t\, d\lambda = \begin{cases} 0 & (t > r) \\ (r^2 - t^2)^{-1/2} & (0 \leqslant t < r) \end{cases} \tag{2.7}$$

$$J_0(\lambda r) = \frac{2}{\pi} \int_0^{1/2\pi} \cos(\lambda r \sin\theta)\, d\theta \tag{2.8}$$

leads to the relation

$$\int_0^r \frac{\varphi(t)\, dt}{\sqrt{r^2 - t^2}} - \frac{2}{\pi} \int_0^{1/2\pi} d\theta \int_0^a \varphi(t)\, dt \int_0^\infty g(\lambda) \cos \lambda t \cos(\lambda r \sin\theta)\, d\lambda = f(r) \tag{2.9}$$

Introducing in the first of the integrals a new integration variable by substituting $t = r \sin\theta$, and denoting by $G(x)$ the Fourier cosine-transform of the function $g(\lambda)$, so that

$$G(x) = \int_0^\infty g(\lambda) \cos \lambda x\, d\lambda \tag{2.10}$$

* All computations of this Section are purely formal, but they can be easily justified, if we assume the existence of a solution of Fredholm's equation (2.12) continuous, together with its derivative, in the closed interval $(0, a)$.

relation (2.9) can be written in the form

$$\int_0^{{}^1/_2\pi}\left(\varphi(r\sin\theta)-\frac{1}{\pi}\int_0^a\varphi(t)\,[G(t+r\sin\theta)+G(t-r\sin\theta)]\,dt\right)d\theta=f(r) \quad (2.11)$$

Putting

$$\varphi(x)-\frac{1}{\pi}\int_0^a\varphi(t)\,[G(t+x)+G(t-x)]\,dt=F(x) \qquad (0\leqslant x\leqslant a) \quad (2.12)$$

we arrive at the integral equation of Schloemilch

$$\int_0^{\pi/2}F(r\sin\theta)\,d\theta=f(r) \qquad (0\leqslant r\leqslant a) \quad (2.13)$$

The continuous solution of this equation is given by formula [4]

$$F(x)=\frac{2}{\pi}\left[f(0)+x\int_0^{{}^1/_2\pi}f'(x\sin\theta)\,d\theta\right] \quad (2.14)$$

After determination of $F(x)$, the equality (2.12) can be considered as an integral equation for the unknown function $\phi(t)$. It follows from the definition (2.10) of the function $G(x)$ that the kernel of the equation is a continuous and symmetrical function of the variables x and t. If it is possible to find a solution of the integral equation (2.12) in the form of a function with a continuous derivative, then the formulas (2.5), (2.2) and (2.1) give a complete solution of the contact problem under consideration*.

It should be noted that many quantities, of interest in applications, can be expressed immediately in terms of the function $\phi(t)$ under omission of the intermediate formulas, which is particularly advantageous for numerical methods of solving the equation (2.12). Thus, for instance, using formulas (1.4) and (2.6), we easily obtain the simple formula for the distribution of the normal stresses under the punch:

$$[\sigma_z]_{z=0}=\int_r^a\frac{\varphi'(t)}{\sqrt{t^2-r^2}}\,dt-\frac{\varphi(a)}{\sqrt{a^2-r^2}} \qquad \text{when } r<a \quad (2.15)$$

Particularly simple is the expression obtained for the magnitude of the total pressure of the stamp on the layer, numerically equal to the applied force P. Integrating (2.15) along the area of the circle of radius a, we find

$$P=2\pi\int_0^a\varphi(t)\,dt \quad (2.16)$$

* If $F'(x)$ is a continuous function, then it is sufficient to require the existence of a continuous solution of the equation (2.12).

This is an equation for the determination of the magnitude w_0 of the displacement of the stamp corresponding to a given force P.

In the case of a stamp with non-plane base in non-complete penetration we derive from formula (2.15) the additional equation

$$\varphi(a) = 0 \tag{2.17}$$

which follows from the condition of continuity of normal stresses in the plane $z = 0$. In this case the relations (2.16) and $\phi(a) = 0$ are to be used for determination of the displacement w_0 of the stamp and of the radius a of the contact area.

3. For a stamp with a plane base

$$\chi(r) = 0, \qquad f(r) = \frac{\mu w_0}{1-\nu}$$

and the integral equation (2.17) assumes the form

$$\varphi(x) - \frac{1}{\pi}\int_0^a \varphi(t)\,[G(t+x) + G(t-x)]\,dt = \frac{2\mu w_0}{\pi(1-\nu)} \qquad (0 \leqslant x \leqslant a) \tag{3.1}$$

In further computations it is convenient to use dimensionless quantities by introducing

$$\frac{x}{a} = \xi, \qquad \frac{t}{a} = \tau, \qquad \varphi(x) = \frac{2\mu w_0}{\pi(1-\nu)}\,\omega(\xi) \tag{3.2}$$

Equation (3.1) assumes then the form

$$\omega(\xi) = 1 + \frac{1}{\pi}\int_0^1 [K(\tau+\xi) + K(\tau-\xi)]\,\omega(\tau)\,d\tau \qquad (0 \leqslant \xi \leqslant 1) \tag{3.3}$$

where

$$K(u) = p\int_0^\infty \frac{\alpha + e^{-\alpha}\operatorname{sh}\alpha}{\alpha + \operatorname{sh}\alpha\operatorname{ch}\alpha}\cos\alpha p u\,d\alpha \qquad \left(p = \frac{a}{h}\right) \tag{3.4}$$

In the case of small values of the parameter p the solution of the equation (3.3) can be represented by an expansion into powers of that parameter. In the general case it is necessary to use numerical methods; this requires in the first place a tabulation of the function $K(u)$ in the interval $0 < u < 2$. Replacing, furthermore, the value of the integral in (3.3) by its approximation obtained by use of the trapezium formula, or another quadrature formula, we reduce the problem of determining $\omega(r)$ to that of solving a system of linear equations, which makes it possible to set up a table of numerical values of this function with a necessary degree of accuracy.

The formula (2.16), which gives the required connection between the displacement w_0 of the stamp and the force P, assumes in dimensionless variables the form

$$\varkappa = \frac{P(1-\nu)}{4a\mu w_0} = \int_0^1 \omega(\tau)\,d\tau \tag{3.5}$$

Therefore, having a table of the numerical values of the function $\omega(\tau)$, we will be able without difficulties to compute the corresponding values of the coefficient κ.

Following this scheme, computations were carried out for five values of the parameter p, from $p = 0$ to $p = 2$ at intervals equal to 0.5. The value $p = 0$ corresponds to the case of a stamp acting on an elastic half-space. In this case $K(u) \equiv 0$, $\omega(\tau) \equiv 1$, $\kappa = 1$. The results of the computations are given in Tables 1 to 3. It should be noted that Table 1 can be used also in the study of the contact problem for a stamp with non-plane base.

TABLE 1. Numerical Values of the Kernel $K(u)$

u	$p = 0.5$	$p = 1.0$	$p = 1.5$	$p = 2.0$
0	0.5837	1.1674	1.7511	2.3349
0.1	0.5828	1.1596	1.7248	2.2728
0.2	0.5798	1.1364	1.6486	2.0990
0.3	0.5750	1.0991	1.5310	1.8448
0.4	0.5682	1.0495	1.3836	1.5513
0.5	0.5598	0.9897	1.2196	1.2565
0.6	0.5496	0.9224	1.0515	0.9875
0.7	0.5378	0.8502	0.8894	0.7591
0.8	0.5248	0.7757	0.7406	0.5753
0.9	0.5104	0.7010	0.6090	0.4336
1.0	0.4948	0.6282	0.4962	0.3279
1.1	0.4784	0.5587	0.4021	0.2512
1.2	0.4612	0.4937	0.3252	0.1962
1.3	0.4434	0.4339	0.2635	0.1571
1.4	0.4251	0.3795	0.2148	0.1291
1.5	0.4065	0.3308	0.1767	0.1088
1.6	0.3878	0.2877	0.1472	0.0939
1.7	0.3691	0.2498	0.1243	0.0826
1.8	0.3505	0.2168	0.1065	0.0737
1.9	0.3322	0.1883	0.0925	0.0663
2.0	0.3141	0.1640	0.0816	0.0602

TABLE 2. Numerical Values of the Kernel $\omega(\tau)$

τ	$p = 0.5$	$p = 1.0$	$p = 1.5$	$p = 2.0$
0	1.530	2.353	3.320	4.321
0.1	1.529	2.348	3.308	4.303
0.2	1.527	2.333	3.272	4.246
0.3	1.523	2.307	3.212	4.153
0.4	1.518	2.272	3.129	4.022
0.5	1.511	2.228	3.025	3.854
0.6	1.503	2.176	2.902	3.650
0.7	1.494	2.118	2.763	3.416
0.8	1.484	2.054	2.611	3.157
0.9	1.472	1.986	2.454	2.884
1.0	1.460	1.915	2.290	2.608

TABLE 3.

p	0	0.5	1.0	1.5	2.0
$\varkappa$	1	1.51	2.20	2.95	3.72

4. The general formulas derived in Section 2 permit to consider the limiting case $h \to \infty$, investigated in publications of many other authors (see e.g. [1] and [3]). For this case the function $G(x) \equiv 0$ and the formula (2.12) gives an explicit expression for the required function, namely

$$\varphi(x) = F(x) = \frac{2}{\pi}\left[f(0) + x\int_0^{\infty} f\ (x\sin\theta)\,d\theta\right] \tag{4.1}$$

where $f(r)$ is defined by (2.4).

The required connection between the displacement w_0 of the stamp and the force P is determined immediately from (4.1) and (2.16) and we find

$$P = \frac{4a\mu}{1-\nu}\left[w_0 - \int_0^{\frac{1}{2}\pi} \chi(a\sin\theta)\sin\theta\,d\theta\right] \tag{4.2}$$

In the case of non-complete penetration of a stamp with non-plane base we also have to consider the additional equation (2.17), which in this case assumes the form

$$\frac{w_0}{a} = \int_0^{\pi/2} \chi'(a\sin\theta)\,d\theta \tag{4.3}$$

From equations (4.2) and (4.3) we have to find the magnitudes of the displacement w_0 of the stamp and of the radius a of the contact area. Furthermore, the formulas (4.2) and (4.3) permit also to determine, for a given radius a_0 of the stamp, the limiting value of the force P_0, starting from which a complete penetration takes place.

$$P_0 = \frac{4a_0\mu}{1-\nu}\int_0^{\pi/2}\left[a_0\chi'(a_0\sin\theta) - \sin\theta\chi(a_0\sin\theta)\right]d\theta \tag{4.4}$$

In conclusion, we give also the formula for the normal stresses in the contact area ($r < a$):

$$[\sigma_z]_{z=0} = -\frac{2\mu}{\pi(1-\nu)}\left\{\left[w_0 - a\int_0^{\frac{1}{2}\pi}\chi'(a\sin\theta)\,d\theta\right]\frac{1}{\sqrt{a^2-r^2}} + \right.$$

$$\left. + \int_r^a \frac{d^2}{dt^2}\left[\int_0^{\frac{1}{2}\pi} t\sin\theta\chi(t\sin\theta)\,d\theta\right]\frac{dt}{\sqrt{t^2-r^2}}\right\} \tag{4.5}$$

BIBLIOGRAPHY

1. Galin, L.A., *Kontaktnye zadachi teorii uprugosti (Contact Problems of the Theory of Elasticity)*. GTTI, 1953.

2. Lebedev, N.N., Raspredelenie elektrichestva na tonkom paraboloidal'nom segmente (Distribution of elektricity on a thin paraboloidal segment). *Dokl. Akad. Nauk SSSR* Vol. 114, No. 3, 1957.

3. Lur'e A.I., *Prostranstvennye zadachi teorii uprugosti (Three-dimensional Problems of the Theory of Elasticity)*. GTTI, 1955.

4. Whittaker, E.T. and Watson, G.N. *Kurs Sovremennogo Analiza (Modern Analysis)*. Part I. GTTI, 1933.

Translated by I.M.